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DISPERSION RELATIONS AND WAVE OPERATORS IN SELF-SIMILAR QUASI-CONTINUOUS LINEAR CHAINS

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1 Abstract

We construct self-similar functions and linear operators to deduce a self-similar variant of the Laplacian operator and of the D'Alembertian wave operator. The exigence of self-similarity as a symmetry property requires the introduction of non-local particle-particle interactions. We derive a self-similar linear wave operator describing the dynamics of a quasi-continuous linear chain of infinite length with a spatially self-similar distribution of nonlocal inter-particle springs. The self-similarity of the nonlocal harmonic particle-particle interactions results in a dispersion relation of the form of a Weierstrass-Mandelbrot function which exhibits self-similar and fractal features. We also derive a continuum approximation which relates the self-similar Laplacian to fractional integrals and yields in the low-frequency regime a power law frequency-dependence of the oscillator density.

Keywords: Self-similarity, self-similar functions, affine transformations, Weierstrass-Mandelbrot function, fractal functions, fractals, power laws, fractional integrals.

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2 Introduction

In the seventies of the last century the development of the *Fractal Geometry* by Mandelbrot [1] launched a scientific revolution. However, the mathematical roots of this discipline originate much earlier in the 19^{th} century [2]. The superior electromagnetic properties of "*fractal antennae*" have been known already for a while [3, 4]. More recently one found by means of numerical simulations that fractal gaskets such as the Sierpinski gasket reveal interesting vibrational properties [5]. Meanwhile physical problems in fractal and self-similar structures or media become more and more a subject of interest also in analytical mechanics and engineering science. This is true in statics and dynamics. However technological exploitations of effects based on self-similarity and "fractality" are still very limited due to a lack of fundamental understanding of the role of the self-similar symmetry. An improved understanding could raise an enormous new field for basic research and applications in a wide range of disciplines including fluid mechanics and the mechanics of granular media and solids. Some initial steps have been performed (see papers [5, 6, 7, 8, 9, 10] and the references therein). However a generally accepted "fractal mechanics" has yet to be developed. Therefore, it is highly desirable to develop sufficiently simple models which are on the one hand accessible to a mathematical-analytical framework and on the other hand which capture the essential features imposed by self-similar scale invariant symmetry. The goal of this demonstration is to develop such a model.

Several significant contributions of fractal and self-similar chains and lattices have been presented [13, 14, 15, 16]. In these papers problems on *discrete* lattices with fractal features are addressed. Closed form solutions for the dynamic Green's function and the vibrational spectrum of a linear chain with spatially exponential properties are developed in a recent paper [11]. A similar fractal type of linear chain as in the present paper has been considered very recently by Tarasov [7]. Unlike in the present paper the chain considered by Tarasov in [7] is *discrete*, i.e. there remains a characteristic length scale which is given by the next-neighbor distance of the particles.

In contrast to all these works we analyze in the present paper vibrational properties in a *quasi-continuous* linear chain with (in the self-similar limiting case) infinitesimal lattice spacing with a non-local spatially self-similar distribution of power-law-scaled harmonic inter-particle interactions (springs). In this way we avoid the appearance of a characteristic length scale in our chain model. It seems there are analogue situations in turbulence [17] and other areas where the present interdisciplinary approach could be useful.

Our demonstration is organized as follows: § 3 is devoted to the construction of self-similar functions and operators where a self-similar variant of the Laplacian is deduced. This Laplacian gets his physical justification in § 4. It is further shown in § 3 that in a continuum approximation this Laplacian takes the form of fractional integrals. In § 4 we consider a self-similar quasi-continuous linear chain with self-similar harmonic interactions. The equation of motion of this chain takes the form of a self-similar wave equation containing the self-similar Laplacian defined in § 3 leading to a dispersion relation having the form of the Weierstrass-Mandelbrot function which is a self-similar and for a certain parameter range also a fractal function.

3 Construction of self-similar functions and linear operators

In this paragraph we define the term "self-similarity" with respect to functions and operators. We call a scalar function $\phi(h)$ *exact self-similar* with respect to variable h if the condition

$$\phi(Nh) = \Lambda\phi(h) \tag{1}$$

is satisfied for all values $h > 0$ of the scalar variable h. We call (1) the "affine problem"[1] where N is a fixed parameter and $\Lambda = N^{\delta}$ represents a continuous set of admissible eigenvalues. The band of

[1]where we restrict here to affine transformations $h' = Nh + c$ with $c = 0$.

admissible $\delta = \frac{\ln \Lambda}{\ln N}$ is to be determined. A function $\phi(h)$ satisfying (1) for a certain N and admissible $\Lambda = N^\delta$ represents an unknown "solution" to the affine problem of the form $\phi_{N,\delta}(h)$ and is to be determined.

As we will see below for a given N solutions $\phi(h)$ *exist* only in a certain range of admissible Λ. From the definition of the problem follows that if $\phi(h)$ is a solution of (1) it is also a solution of $\phi(N^s h) = \Lambda^s \phi(h)$ where $s \in \mathbb{Z}$ is discrete and can take all positive and negative integers including zero. We emphasize that non-integer s are not admitted. The discrete set of pairs Λ^s, N^s are for all $s \in \mathbb{Z}$ related by a power law with the same power δ, i.e. $\Lambda = N^\delta$ hence $\Lambda^s = (N^s)^\delta$. By replacing Λ and N by Λ^{-1} and N^{-1} in (1) defines the identical problem. Hence we can restrict our considerations on fixed values of $N > 1$.

We can consider the affine problem (1) as the eigenvalue problem for a linear operator $\hat{A}_N$ with a certain given fixed parameter N and eigenfunctions $\phi(h)$ to be determined which correspond to an *admissible* range of eigenvalues $\Lambda = N^\delta$ (or equivalently to an admissible range of exponent $\delta = \ln \Lambda / \ln N$). For a function $f(x, h)$ we denote by $\hat{A}_N(h) f(x, h) =: f(x, Nh)$ when the affine transformation is only performed with respect to variable h.

We assume $\Lambda, N \in \mathbb{R}$ for physical reasons without too much loss of generality to be real and positive. For our convenience we define the "affine" operator $\hat{A}_N$ as follows

$$\hat{A}_N f(h) =: f(Nh) \tag{2}$$

It is easily verified that the affine operator $\hat{A}_N$ is *linear*, i.e.

$$\hat{A}_N \left(c_1 f_1(h) + c_2 f_2(h)\right) = c_1 f_1(Nh) + c_2 f_2(Nh) \tag{3}$$

and

$$\hat{A}_N^s f(h) = f(N^s h), \qquad\qquad s = 0 \pm 1, \pm 2, .. \pm \infty \tag{4}$$

We can define affine operator functions for any smooth function $g(\tau)$ that can be expanded into a Maclaurin series as

$$g(\tau) = \sum_{s=0}^{\infty} a_s \tau^s \tag{5}$$

We define an affine operator function in the form

$$g(\xi \hat{A}_N) = \sum_{s=0}^{\infty} a_s \xi^s \hat{A}_N^s \tag{6}$$

where ξ denotes a scalar parameter. The operator function which is defined by (6) acts on a function $f(h)$ as follows

$$g(\xi \hat{A}_N) f(h) = \sum_{s=0}^{\infty} a_s \xi^s f(N^s h) \tag{7}$$

where relation (4) with expansion (6) has been used. The convergence of series (7) has to be verified for a function $f(h)$ to be admissible. An explicit representation of the affine operator $\hat{A}_N$ can be obtained when we write $f(h) = f(e^{\ln h}) = \bar{f}(\ln h)$ to arrive at

$$\hat{A}_N(h) = e^{\ln N \frac{\mathrm{d}}{\mathrm{d(\ln h)}}} \tag{8}$$

This relation is immediately verified in view of

$$\hat{A}_N(h) f(h) = e^{\ln N \frac{\mathrm{d}}{\mathrm{d(\ln h)}}} f(e^{\ln h}) = f(e^{\ln h + \ln N}) = f(Nh) \tag{9}$$

With this machinery we are now able to construct self-similar functions and operators.

3.1 Construction of self-similar functions

A self-similar function solving problem (1) is formally given by the series

$$\phi(h) = \sum_{s=-\infty}^{\infty} \Lambda^{-s} \hat{A}_N^s f(h) = \sum_{s=-\infty}^{\infty} \Lambda^{-s} f(N^s h) \tag{10}$$

for any function $f(h)$ for which the series (10) is uniformly convergent for all h. We introduce the self-similar operator

$$\hat{T}_N = \sum_{s=-\infty}^{\infty} \Lambda^{-s} \hat{A}_N^s \tag{11}$$

that fulfils formally the condition of self-similarity $\hat{A}_N \hat{T}_N = \Lambda \hat{T}_N$ and hence (10) solves the affine problem (1). In view of the symmetry with respect to inversion of the sign of s in (10) and (11) we can restrict ourselves to $N > 1$ ($N, \Lambda \in \mathbb{R}$) without any loss of generality[2]: Let us look for admissible functions $f(t)$ for which (10) is convergent. To this end we have to demand simultaneous convergence of the partial sums over positive and negative s. Let us assume that (where we can confine ourselves to $t > 0$)

$$\lim_{t \to 0} f(t) = a_0 \, t^{\alpha} \tag{12}$$

For $t \to \infty$ we have to demand that $|f(t)|$ increases not stronger than a power of t, i.e.

$$\lim_{t \to \infty} f(t) = c_{\infty} \, t^{\beta} \tag{13}$$

with a_0, c_∞ denoting constants. Both exponents $\alpha, \beta \in \mathbb{R}$ are allowed to take positive or negative values which do not need to be integers. A brief consideration of partial sums yields the following requirements for $\Lambda = N^{\delta}$, namely: Summation over $s < 0$ in (10) requires absolute convergence of a geometrical series leading to the condition for its argument $\Lambda N^{-\alpha} < 1$. That is we have to demand $\delta < \alpha$. The partial sum over $s > 0$ requires absolute convergence of a geometrical series leading to the condition for its argument $\Lambda^{-1} N^{\beta} < 1$ which corresponds to $\delta > \beta$. Both conditions are simultaneously met if

$$N^{\beta} < \Lambda = N^{\delta} < N^{\alpha} \tag{14}$$

or equivalently

$$\beta < \delta = \frac{\ln \Lambda}{\ln N} < \alpha \tag{15}$$

Relations (14) and (15) require additionally $\beta < \alpha$, that is only functions $f(t)$ with the behaviour (12) and (13) with $\beta < \alpha$ are *admissible* in (10). The case $\beta = 0$ includes for instance certain bounded functions $|f(t)| < M$ such as some periodic functions.

3.2 A self-similar analogue to the Laplace operator

In the sprit of (10) and (11) we construct an exactly self-similar function from the second difference according to

$$\phi(x, h) = \hat{T}_N(h) \left(u(x+h) + u(x-h) - 2u(x) \right) \tag{16}$$

[2] We also can exclude the trivial case $N = 1$.

where $u(..)$ denotes an arbitrary smooth continuous field variable and $\hat{T}_N(h)$ expresses that the affine operator $\hat{A}_N(h)$ acts only on the dependence on h, that is $\hat{A}_N(h)v(x,h) = v(x,Nh)$. We have with $\xi = \Lambda^{-1}$ the expression

$$\phi(x,h) = \sum_{s=-\infty}^{\infty} \xi^s \left\{ u(x+N^s h) + u(x-N^s h) - 2u(x) \right\} \tag{17}$$

which is a self-similar function with respect to its dependence on h with $\hat{A}_N(h)\phi(x,h) = \phi(x,Nh) = \xi^{-1}\phi(x,h)$ but a regular function with respect to x. The function $\phi(x,h)$ exists if the series (17) is convergent. Let us assume that $u(x)$ is a smooth function with a convergent Taylor series for any h. Then we have with $u(x\pm h) = e^{\pm h\frac{d}{dx}}u(x)$ and $u(x+h)+u(x-h)-2u(x) = \left(e^{h\frac{d}{dx}} + e^{-h\frac{d}{dx}} - 2\right)u(x)$ which can be written as

$$u(x+h)+u(x-h)-2u(x) = 4\sinh^2\left(\frac{h}{2}\frac{d}{dx}\right)u(x) = h^2\frac{d^2}{dx^2}u(x) + orders\ h^{\geq 4} \tag{18}$$

thus $\alpha = 2$ in criteria (12) is met. If we demand $u(x)$ being Fourier transformable we have as necessary condition that

$$\int_{-\infty}^{\infty} |u(x)|\, \mathrm{d}x < \infty \tag{19}$$

exists. This is true if $|u(t)|$ tends to zero as $t \to \pm\infty$ as $|t|^\beta$ where $\beta < -1$. We have then the condition that

$$\beta < 0 < \delta = -\frac{\ln \xi}{\ln N} < \alpha = 2 \tag{20}$$

We will see below that only $\delta > 0$ is *physically admissible*, i.e. compatible with harmonic particle-particle interactions which decrease with increasing particle-particle distance.

The 1D Laplacian Δ_1 is defined by

$$\Delta_1 u(x) = \frac{d^2}{dx^2}u(x) = \lim_{\tau\to 0} \frac{(u(x+\tau)+u(x-\tau)-2u(x))}{\tau^2} \tag{21}$$

Let us now define a self-similar analogue to the 1D Laplacian (21) where we put with $\xi = N^{-\delta}$

$$\Delta_{(\delta,N,\tau)}u(x) =: const \lim_{\tau\to 0} \tau^{-\lambda}\phi(x,\tau) \tag{22}$$

$$= const\ \lim_{\tau\to 0} \tau^{-\lambda} \sum_{s=-\infty}^{\infty} \xi^s \left(u(x+N^s\tau) + u(x-N^s\tau) - 2u(x)\right) \tag{23}$$

where we have introduced a renormalisation-multiplier $\tau^{-\lambda}$ with the unknown power λ to be determined such that the limiting case is finite. The constant factor *const* indicates that there is a certain arbitrariness in this definition and will be chosen conveniently. Let us consider the limit $\tau \to 0$ by the special sequence $\tau_n = N^{-n}h$ with $n \to \infty$ and h being constant. Unlike in the 1D case (21), the result of this limiting process depends crucially on the choice of the sequence τ_n. We see here that the self-similar Laplacian cannot be defined uniquely as in the 1D case. We have (by putting in (22) $const = h^\lambda$)

$$\Delta_{(\delta,N,h)}u(x) = \lim_{n\to\infty} N^{\lambda n}\xi^n \sum_{s=-\infty}^{\infty} \xi^{s-n}\left(u(x+N^{s-n}h) + u(x-N^{s-n}h) - 2u(x)\right) \tag{24}$$

which assumes by replacing $s - n \to s$ the form

$$\Delta_{(\delta,N,h)}u(x) = \phi(x,h) \lim_{n\to\infty} N^{-(\delta-\lambda)n} \tag{25}$$

which is only finite and nonzero if $\lambda = \delta$. The "Laplacian" can then be defined simply by

$$\Delta_{(\delta,N,h)} u(x) =: \lim_{n\to\infty} N^{\delta n} \phi(x, N^{-n}h) = \phi(x,h) \tag{26}$$

or by using (16) and (18) we can simply write[3]

$$\Delta_{(\delta,N,h)} = 4\hat{T}_N(h) \sinh^2\left(\frac{h}{2}\frac{\partial}{\partial x}\right) = 4 \sum_{s=-\infty}^{\infty} N^{-\delta s} \sinh^2\left(\frac{N^s h}{2}\frac{\partial}{\partial x}\right) \tag{27}$$

where $\hat{T}_N(h)$ is the self-similar operator defined in (11). The self-similar analogue of Laplace operator defined by (27) depends on the parameters δ, N, h. We furthermore observe the self-similarity of Laplacian (27) with respect to its dependence on h, namely

$$\Delta_{(\delta,N,Nh)} = N^{\delta} \Delta_{(\delta,N,h)} \tag{28}$$

3.3 Continuum approximation - link to fractional integrals

For numerical evaluations it may be convenient to utilize a continuum approximation of the self-similar Laplacian (27). To this end we put $N = 1+\epsilon$ (with $0 < \epsilon \ll 1$ thus $\epsilon \approx \ln N$) where ϵ is assumed to be "small" and $s\epsilon = v$ such that $dv \approx \epsilon$ and $N^s = (1+\epsilon)^{\frac{v}{\epsilon}} \approx e^v$. In this approximation $N^s \approx e^v$ becomes a (quasi)-continuous variable when s runs through $s \in \mathbb{Z}$. Then we can write (10) in the form

$$\phi(h) = \sum_{s=-\infty}^{\infty} N^{-s\delta} f(N^s h) \approx \frac{1}{\epsilon} \int_{-\infty}^{\infty} e^{-\delta v} f(he^v) \mathrm{d}v \tag{29}$$

which can be further written with $he^v = \tau$ ($h > 0$) and $\frac{\mathrm{d}\tau}{\tau} = \mathrm{d}v$ and $\tau(v \to -\infty) = 0$ and $\tau(v \to \infty) = \infty$ as

$$\phi(h) \approx \frac{h^\delta}{\epsilon} \int_0^\infty \frac{f(\tau)}{\tau^{1+\delta}} \mathrm{d}\tau \tag{30}$$

In this continuum approximation the function $\phi(h)$ obeys the same scaling behaviour as (10), namely $\phi(h\lambda) = \lambda^\delta \phi(h)$ but in contrast to (10) λ can assume any continuous positive value. This is due to the fact that (30) is holding for $N = 1+\epsilon$ with sufficiently small $\epsilon > 0$ since in this limiting case there exists for any continuous value $\lambda > 0$ an $m \in \mathbb{Z}$ such that $N^m \approx \lambda$. The existence requirement for integral (30) leads to the same requirements for $f(t)$ as in (10), namely inequality (15). Application of the approximate relation (30) to Laplacian (27) yields

$$\Delta_{(\delta,\epsilon,h)} u(x) \approx \frac{h^\delta}{\epsilon} \int_0^\infty \frac{(u(x-\tau) + u(x+\tau) - 2u(x))}{\tau^{1+\delta}} \mathrm{d}\tau \tag{31}$$

where this integral exists for $\beta < 0 < \delta < 2$ and $\beta < -1$ because the required existence of integral (19) and relation (18). By performing two partial integrations and by taking into account the vanishing boundary terms at $\tau = 0$ and $\tau = \infty$ for $0 < \delta < 2$, we can re-write (31) in the form of a convolution of the conventional 1D Laplacian $\frac{d^2u}{dx^2}(x)$, namely

$$\Delta_{(\delta,\epsilon,h)} u(x) \approx \int_{-\infty}^{\infty} g(|x-\tau|) \frac{d^2u}{d\tau^2}(\tau) \mathrm{d}\tau \tag{32}$$

with the kernel

[3] We have to replace $\frac{d}{dx} \to \frac{\partial}{\partial x}$ if the Laplacian acts on a field $u(x,t)$ as in Sec. 4.

$$g(|x|) = \frac{h^{\delta}}{\delta(\delta-1)\epsilon} |x|^{1-\delta}, \qquad \delta \neq 1 \tag{33}$$

where $0 < \delta < 2$ and $g(|x|) = -\frac{h}{\epsilon} \ln |x|$ for $\delta = 1$. We can further write for $\delta \neq 1$ (32) in terms of *fractional integrals*

$$\Delta_{(\delta=2-D,\epsilon,h)} u(x) \approx \frac{h^{2-D}}{\epsilon} \frac{\Gamma(D)}{(D-1)(D-2)} \left(\mathcal{D}^{-D}_{-\infty,x} + (-1)^D \mathcal{D}^{-D}_{\infty,x}\right) \Delta_1 u(x) \tag{34}$$

where $\Delta_1 u(x) = \frac{d^2}{dx^2} u(x)$ denotes the conventional 1D-Laplacian and $D = 2-\delta > 0$ which is positive in the admissible range of $0 < \delta < 2$. For $0 < \delta < 1$ the quantity D can be identified with the estimated fractal dimension of the fractal dispersion relation of the Laplacian [18] which is deduced in the next section. In (34) we have introduced the Riemann-Liouville fractional integral $\mathcal{D}^{-D}_{a,x}$ which is defined by(e.g. [19, 20])

$$\mathcal{D}^{-D}_{a,x} v(x) = \frac{1}{\Gamma(D)} \int_a^x (x-\tau)^{D-1} v(\tau) \mathrm{d}\tau \tag{35}$$

where $\Gamma(D)$ denotes the Γ-function which represents the generalization of the factorial function to non-integer $D > 0$. The Γ-function is defined as

$$\Gamma(D) = \int_0^{\infty} \tau^{D-1} e^{-\tau} \mathrm{d}\tau \,, \qquad D > 0 \tag{36}$$

For positive integers $D > 0$ the Γ-function reproduces the factorial-function $\Gamma(D) = (D-1)!$ with $D = 1, 2, ..\infty$.

4 The physical chain model

We consider an infinitely long quasi-continuous linear chain of identical particles. Any space-point x corresponds to a "material point" or particle. The mass density of particles is assumed to be spatially homogeneous and equal to one for any space point x. Any particle is associated with one degree of freedom which is represented by the displacement field $u(x,t)$ where x is its spatial (Lagrangian) coordinate and t indicates time. In this sense we consider a quasi continuous spatial distribution of particles. Any particle at space-point x is non-locally connected by harmonic springs of strength ξ^s to particles located at $x \pm N^s h$, where $N > 1$ and $N \in \mathbb{R}$ is not necessarily integer, $h > 0$, and $s = 0, \pm 1, \pm 2, .. \pm \infty$. The requirement of decreasing spring constants with increasing particle-particle distance leads to the requirement that $\xi = N^{-\delta} < 1$ $(N > 1)$ i.e. only chains with $\delta > 0$ are physically admissible. In order to get exact self-similarity we avoid the notion of "next-neighbour particles" in our chain which would be equivalent to the introduction of an internal length scale (the next neighbour distance). To admit particle interactions over arbitrarily close distances $N^s h \to 0$ $(s \to -\infty,\ h = const)$ our chain has to be *quasi-continuous*. This is the principal difference to the *discrete* chain considered recently by Tarasov [7] which is discrete and not self-similar.

The Hamiltonian which describes our chain can be written as

$$H = \frac{1}{2} \int_{-\infty}^{\infty} \left(\dot{u}^2(x,t) + \mathcal{V}(x,t,h)\right) \mathrm{d}x \tag{37}$$

In the spirit of (10) the elastic energy density $\mathcal{V}(x,t,h)$ is assumed to be constructed self-similarly, namely[4]

$$\mathcal{V}(x,t,h)=\frac{1}{2}\hat{T}_N(h)\left[(u(x,t)-u(x+h,t))^2+(u(x,t)-u(x-h,t))^2\right] \tag{38}$$

where $\hat{T}_N(h)$ is the self-similar operator (11) with $\xi=\Lambda^{-1}=N^{-\delta}$ to arrive at

$$\mathcal{V}(x,t,h)=\frac{1}{2}\sum_{s=-\infty}^{\infty}\xi^s\left[(u(x,t)-u(x+hN^s,t))^2+(u(x,t)-u(x-hN^s,t))^2\right] \tag{39}$$

The elastic energy density $\mathcal{V}(x,t,h)$ fulfills the condition of self-similarity with respect to h, namely

$$\hat{A}_N(h)\mathcal{V}(x,t,h)=\mathcal{V}(x,t,Nh)=\xi^{-1}\mathcal{V}(x,t,h) \tag{40}$$

We have to demand in our physical model that the energy is finite, i.e. (39) needs to be convergent which yields $\alpha=2$ as for the Laplacian (17). To determine β we have to demand that $u(x,t)$ is a Fourier transformable field[5]. Thus we have to have an asymptotic behaviour of $|u(x\pm\tau,t)|\rightarrow 0$ as τ^β where $\beta<-1$ as $\tau\rightarrow\infty$. From this follows $|u(x,t)-u(x\pm\tau,t)|^2$ behaves then as $|u(x,t)|^2$. Hence, the elastic energy density (39) is finite if

$$0<\delta<2 \tag{41}$$

where $\beta<-1$.

This inequality determines the range of the admissible values of δ in order to achieve convergence. The equation of motion is obtained from

$$\frac{\partial^2 u}{\partial t^2}=-\frac{\delta H}{\delta u} \tag{42}$$

(where $\delta./\delta u$ stands for a functional derivative) to arrive at

$$\frac{\partial^2 u}{\partial t^2} = -\sum_{s=-\infty}^{\infty}\xi^s\left(2u(x,t)-u(x+hN^s,t)-u(x-hN^s,t)\right) \tag{43}$$

$$\frac{\partial^2 u}{\partial t^2} = \Delta_{(\delta,N,h)}u(x,t) \tag{44}$$

with the self-similar Laplacian $\Delta_{(\delta,N,h)}$ of equation (27). As the elastic energy density (39) the equation of motion is convergent for δ being in the interval (41) where $\beta<-1$. We can re-write (44) in the compact form of a wave equation

$$\Box_{(\delta,N,h)}u(x,t)=0 \tag{45}$$

where $\Box_{(\delta,N,h)}$ is the *self-similar analogue of the d'Alembertian wave operator* having the form

$$\Box_{(\delta,N,h)}=\Delta_{(\delta,N,h)}-\frac{\partial^2}{\partial t^2} \tag{46}$$

The d'Alembertian (46) with the Laplacian (27) describes the wave propagation in the self-similar chain with Hamiltonian (37). The present approach seems to be useful as a point of departure to establish a generalized theory of wave propagation in self-similar media.

Now we determine the dispersion relation, which is constituted by the (negative) eigenvalues of the (semi-)negative definite Laplacian (27). To this end we make use of the fact that the displacement

[4]The additional factor 1/2 in the elastic energy avoids double counting.

[5]This assumption defines the (function) space of eigenmodes and corresponds to infinite body boundary conditions.

field $u(x,t)$ is Fourier transformable (guaranteed by choosing $\beta < -1$) and that the exponentials e^{ikx} are eigenfunctions of the self-similar Laplacian (27). We hence write the Fourier integral

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{u}(k,t)e^{ikx}\mathrm{d}k \tag{47}$$

to re-write (44) for the Fourier amplitudes $\tilde{u}(k,t)$ in the form

$$\frac{\partial^2 \tilde{u}}{\partial t^2}(k,t) = -\bar{\omega}^2(k)\,\tilde{u}(k,t) \tag{48}$$

and obtain

$$\omega^2(kh) = 4\sum_{s=-\infty}^{\infty} N^{-\delta s}\sin^2\left(\frac{khN^s}{2}\right) \tag{49}$$

The series (49) describes a *Weierstrass-Mandelbrot function* which is a continuous and for $0 < \delta \leq 1$ a nowhere differentiable function [1, 18]. The Weierstrass-Mandelbrot function (49) fulfills the condition of self-similar symmetry, namely

$$\omega^2(Nkh) = N^{\delta}\,\omega^2(kh) \tag{50}$$

where the interval of convergence of the series of the *Weierstrass-Mandelbrot function* (49) is also given by (41). We emphasize that indeed *only* exponents δ in the interval (41) are *admissible* in Hamiltonian (37) with the elastic energy density (39) in order to have a "well-posed" problem.

It was shown by Hardy [18] that for $\xi N > 1$ and $\xi = N^{-\delta} < 1$ or equivalently for

$$0 < \delta < 1 \tag{51}$$

the Weierstrass-Mandelbrot function of the form (49) is not only self-similar but also a *fractal* curve of (estimated) non-integer fractal (Hausdorff) dimension $D = 2 - \delta > 1$. Figs. 2-4 show dispersion curves $\omega^2(kh)$ for different decreasing values of admissible $0 < \delta < 1$ and increasing fractal dimension D. Fig. 1 corresponds to the non-fractal case ($\delta = 1.2 > 1$). The increase of the fractal dimension from Figs. 2-4 is indicated by the increasingly irregular harsh behaviour of the curves. In Fig. 4 the fractal dimension of the dispersion curve is with $D = 1.9$ already close to the plane-filling dimension 2.

To evaluate (49) approximately it is convenient to replace the series by an integral utilizing a similar substitution as in Sec. 3.3 ($\epsilon \approx \ln N$). By doing so we smoothen the Weierstrass-Mandelbrot function (49). It is important to notice that the resulting approximate dispersion relation is hence differentiable and has not any more a fractal dimension $D > 1$ in the interval (51). For sufficiently "small" $|k|h$ ($h > 0$), i.e. in the long-wave regime we arrive at

$$\omega^2(kh) \approx \frac{(h|k|)^{\delta}}{\epsilon}C \tag{52}$$

which is only finite if $(|k|h)^{\delta}$ is in the order of magnitude of ϵ or smaller. This regime which includes the long-wave limit $k \to 0$ is hence characterized by a power law behaviour $\bar{\omega}(k) \approx Const\,|k|^{\delta/2}$ of the dispersion relation. The constant C introduced in (52) is given by the integral

$$C = 2\int_0^{\infty}\frac{(1-\cos\tau)}{\tau^{1+\delta}}\mathrm{d}\tau \tag{53}$$

which exists for δ being within interval (41).

This approximation holds for "small" $\epsilon \approx \ln N \neq 0$ $(0 < \epsilon \ll 1)$[6] which corresponds to the limiting case that $N^s = e^v$ is continuous. In this limiting case we obtain the oscillator density from [11][7]

$$\rho(\omega) = 2\frac{1}{2\pi}\frac{d|k|}{d\omega} \tag{54}$$

which is normalized such that $\rho(\omega)\mathrm{d}\omega$ counts the number (per unit length) of normal oscillators having frequencies within the interval $[\omega, \omega + \mathrm{d}\omega]$. We obtain then a power law of the form

$$\rho(\omega) = \frac{2}{\pi\delta h}\left(\frac{\epsilon}{C}\right)^{\frac{1}{\delta}} \omega^{\frac{2}{\delta}-1} \tag{55}$$

where δ is restricted within interval (41). We observe hence that the power $\frac{2}{\delta}-1$ is restricted within the range $0 < \frac{2}{\delta} - 1 < \infty$ for $0 < \delta < 2$, especially with always vanishing oscillator density $\rho(\omega \to 0) = 0$.

We emphasize that neither is the dependence on k of the Weierstrass-Mandelbrot function (49) represented by a *continuous* $|k|^\delta$-dependence nor is this function differentiable with respect to k. Application of (54) is hence only justified to be applied to the approximative representation (52) if $0 < \epsilon \ll 1$ thus $N = 1 + \epsilon$ is sufficiently close to 1 so that N^s is a quasi-continuous function when s runs through $s \in \mathbb{Z}$. Hence (54) is not generally applicable to (49) for any arbitrary $N > 1$. We can consider (55) as the low-frequency regime $\omega \to 0$ of the oscillator density holding *only* in the quasi-continuous case $N = 1 + \epsilon$ with $0 < \epsilon \ll 1$.

5 Conclusions

We have depicted how self-similar functions and linear operators can be constructed in a simple manner by utilizing a certain category of conventional "admissible" functions. This approach enables us to construct non-local self-similar analogues to the Laplacian and d'Alembert wave operator. The linear self-similar equation of motion describes the propagation of waves in a quasi-continuous linear chain with harmonic non-local self-similar particle-interactions. The complexity which comes into play by the self-similarity of the non-local interactions is completely captured by the dispersion relations which assume the forms of Weierstrass-Mandelbrot functions (49) exhibiting exact self-similarity and for certain parameter combinations (relation (51)) fractal features. In a continuum approximation the self-similar Laplacian is expressed in terms of fractional integrals (eq. (34)) leading for small k (long-wave limit) to a power-law dispersion relation (eq. (52)) and to a power-law oscillator density (eq. (55)) in the low-frequency regime.

The self-similar wave operator (46) with the Laplacian (27) can be generalized to describe wave propagation in fractal and self-similar structures which are fractal subspaces embedded in Euclidean spaces of 1-3 dimensions. The development of such an approach could be a crucial step towards a better understanding of the dynamics in materials with scale hierarchies of internal structures ("multiscale materials") which may be idealized as fractal and self-similar materials.

We hope to inspire further work and collaborations in this direction to develop appropriate approaches useful for the modelling of static and dynamic problems in self-similar and fractal structures in a wider interdisciplinary context.

[6] $\epsilon = 0$ has to be excluded since it corresponds to $N = 1$.

[7] The additional prefactor "2" takes into account the two branches of the dispersion relation (49) (one for $k < 0$ and one for $k > 0$).

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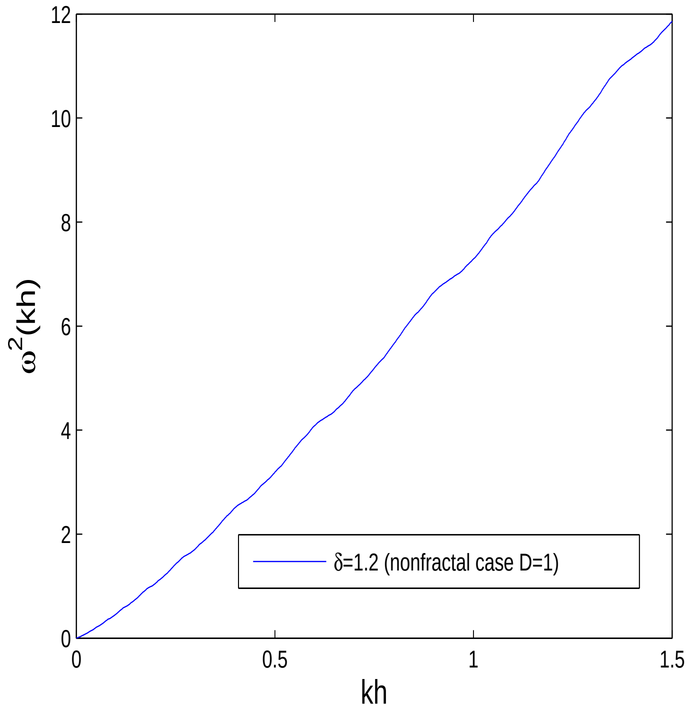

Figure 1: Dispersion relation $\omega^2(kh)$ in arbitrary units for $N = 1.5$ and $\delta = 1.2$

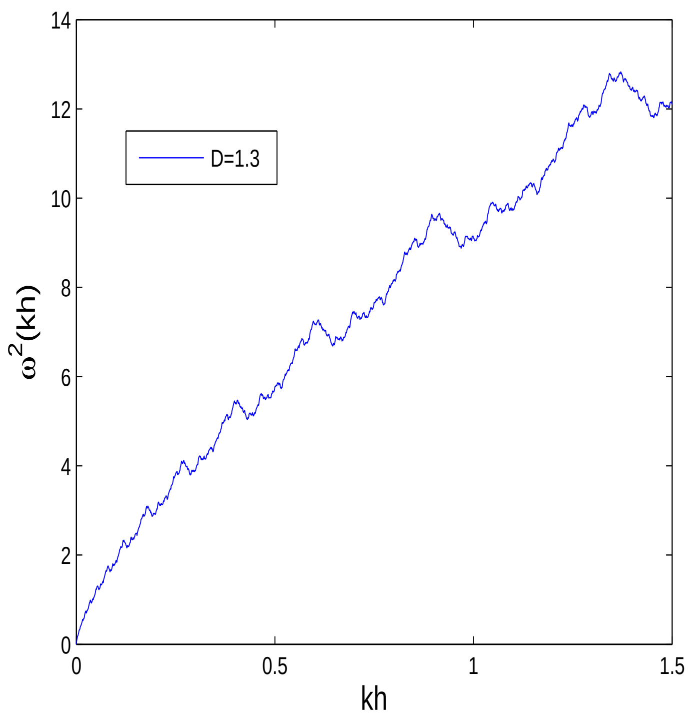

Figure 2: Dispersion relation $\omega^2(kh)$ in arbitrary units for $N = 1.5$ and $\delta = 0.7$

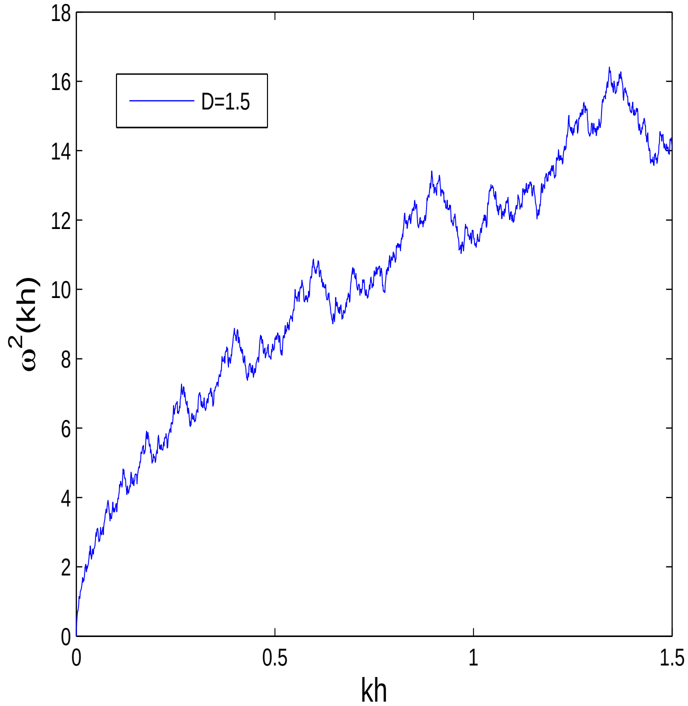

Figure 3: Dispersion relation $\omega^2(kh)$ in arbitrary units for $N = 1.5$ and $\delta = 0.5$

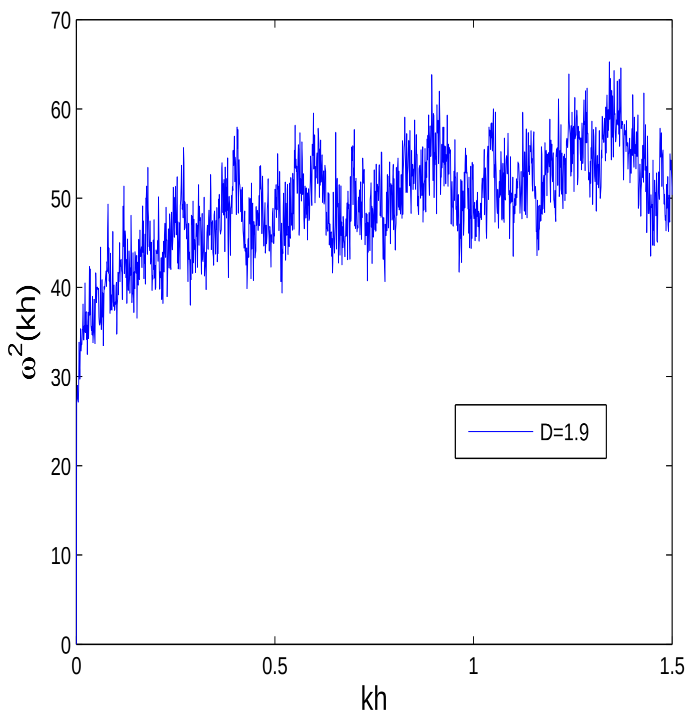

Figure 4: Dispersion relation $\omega^2(kh)$ in arbitrary units for $N = 1.5$ and $\delta = 0.1$